# Game Theory

**Lecture 05**:

Correlated Equilibrium

# Motivating Example I: Bach or Stravinsky

- Two agents, the first a Bach lover and the second a Stravinsky lover, are deciding whether to go to a Bach concert or a Stravinsky concert.

| | $B$ | $S$ |
|---|---|---|
| $b$ | 2,1 | 0,0 |
| $s$ | 0,0 | 1,2 |

"battle of the sexes"

- In this game, there are two pure strategy Nash equilibria:
  - ➢ (b,B) (go to a Bach concert together) and
  - ➢ (s,S) (go to a Stravinsky concert together).
- Either of these behaviors could be considered "unfair:"
  - ➢ The outcome (s, S) is unfair to the first agent, since she prefers to go to the Bach;
  - ➢ The outcome (b,B) is unfair to the second agent, as he prefers to go to the Stravinsky.
- There is one mixed strategy Nash equilibrium in this game:
  - ➢ The first agent plays b with probability 2/3 and
  - ➢ The second agent plays S with probability 2/3.
  - ➢ Here, each agent obtains a utility of 2/3, which is "fair," but is also less than either agent would obtain were either of the pure strategy Nash equilibria to be played!

# Motivating Example I: Bach or Stravinsky

| | $B$ | $S$ |
|---|---|---|
| $b$ | 2,1 | 0,0 |
| $s$ | 0,0 | 1,2 |

- If two people were seeking a fair solution to this game,
  - they might decide to flip a (fair) coin, agreeing in advance that
    - if the coin comes up heads, they both go to the Bach concert,
    - whereas if the coin comes up tails, they both go to the Stravinsky concert.

- This solution is an example of a **correlated equilibrium**, where
  - $\pi$(b,B) = 1/2 and $\pi$(s, S) = 1/2.
- In this example, not only is this solution fair, it is also Pareto-optimal, that is, no agent can be made better off without making some other agent worse off.

# Motivating Example II: Traffic Intersection Game

- Consider a game where two cars arrive at an intersection simultaneously. **In this game, there are three NE:**
  - – two pure: letting only one car cross.
  - – one mixed: both players cross with an extremely small probability $\varepsilon = \frac{1}{101}$ and with probability $\varepsilon^2$ they crash.
  - – The PNE have a payoff of 1.
  - – The MNE is more fair, but has low expected payoff (≈0.0001), and also has a positive chance of a car crash!

| 1 \ 2 | Cross | Stop |
|---|---|---|
| Cross | -100<br>-100 | 0<br>1 |
| Stop | 1<br>0 | 0<br>0 |

*A coordinator (e.g., traffic light) can randomly let one of the two players cross with any probability. The player who is told to stop has 0 payoff, but he knows that attempting to cross will cause a traffic accident.*

# Motivating Example III: Shapley's Game

- Shapley's game is a non-zerosum variant of the famous Rock-Paper-Scissors.
  - ➢ At the unique Nash equilibrium, each agent chooses an action uniformly at random and each agent's expected utility is 1/3.

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $r$ | 0,0 | 0,1 | 1,0 |
| $p$ | 1,0 | 0,0 | 0,1 |
| $s$ | 0,1 | 1,0 | 0,0 |

- However, if a referee selects an action profile uniformly at random from the set:

$$\{(r, P), (r, S), (p, R), (p, S), (s, R), (s, P)\}$$

  - ➢ If the two agents follow the referee's advice, then each agent's expected utility is 1/2.
- Initially, one might think that the referee could select uniformly at random from, say

{(r, P), (p,R)}.

  - ➢ But then, if the first agent were advised to play r, she could infer that:
    - ❖ the second agent was advised to play P, which would motivate her to play s.
    - ❖ If one agent were to cooperate with the referee, the other agent would be motivated to deviate.

# Correlated Equilibrium in One-Shot Games

Consider a (finite) one-shot game as a tuple $\Gamma = \langle N, A, R \rangle$ in which

$N$ is a finite set of $n$ players;

$A = \prod_{i \in N} A_i$, where $A_i$ is player $i$'s finite set of pure actions;

$R : A \to \mathbb{R}^n$, where $R_i(a)$ is player $i$'s reward at action profile $a \in A$.

imagine a referee who selects an action profile $a$ according to some policy $\pi \in \Delta(A)$. It is further assumed that: $\pi$ is common knowledge in the game.

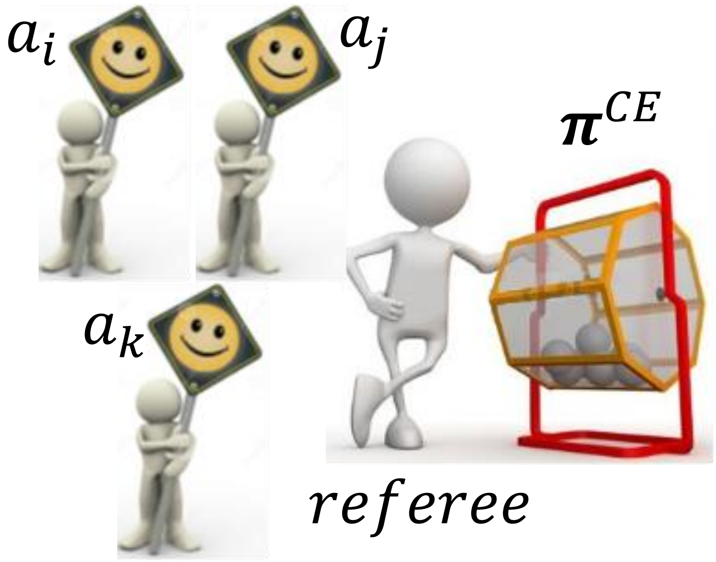
The referee advises player $i$ to follow action $a_i$.

Define

$$\pi(a_i) = \sum_{a_{-i} \in A_{-i}} \pi(a_{-i}, a_i) \text{ and}$$

$$\pi(a_{-i} \mid a_i) = \frac{\pi(a_{-i}, a_i)}{\pi(a_i)} \text{ whenever } \pi(a_i) > 0.$$

Player i's "a posteriori" belief about her opponents' play



**Definition** *Given a one-shot game $\Gamma$, the policy $\pi \in \Delta(A)$ is a* ***correlated equilibrium*** *if, for all $i \in N$, for all $a_i \in A_i$ with $\pi(a_i) > 0$, and for all $a_i' \in A_i$,*

$$\sum_{a_{-i} \in A_{-i}} \pi(a_{-i} \mid a_i) R_i(a_{-i}, a_i) \geq \sum_{a_{-i} \in A_{-i}} \pi(a_{-i} \mid a_i) R_i(a_{-i}, a_i') \qquad (1)$$

If the referee chooses $a$ according to a correlated equilibrium, then the players are motivated to follow his advice, because the expression

$$\sum_{a_{-i} \in A_{-i}} \pi(a_{-i} \mid a_i) R_i(a_{-i}, a_i')$$

computes player $i$'s expected reward for playing $a_i'$ when the referee advises him to play $a_i$.

Equivalently, for all $i \in N$ and for all $a_i, a_i' \in A_i$,

$$\sum_{a_{-i} \in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a_i) \geq \sum_{a_{-i} \in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a_i') \qquad (2)$$

Equation 2 is Equation 1 multiplied by $\pi(a_i)$ and holds trivially whenever $\pi(a_i) = 0$ 

$$\sum_{a_{-i}\in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a_i) \geq \sum_{a_{-i}\in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a_i') \qquad (2)$$

Given a one-shot game $\Gamma$, $R(a_{-i}, a_i)$ is known,

which implies that Equation 2 is a system of linear inequalities,

with $\pi(a_{-i}, a_i)$ unknown.

- ✓ The set of all solutions to a system of linear inequalities is **convex**.
- ✓ Since these inequalities are not strict, this set is also closed.
- ✓ This set is bounded as well, because the set of all policies is bounded.

**Therefore, the set of correlated equilibria is <u>compact</u> and <u>convex</u>.**

# On the geometry of Nash equilibria and correlated equilibria

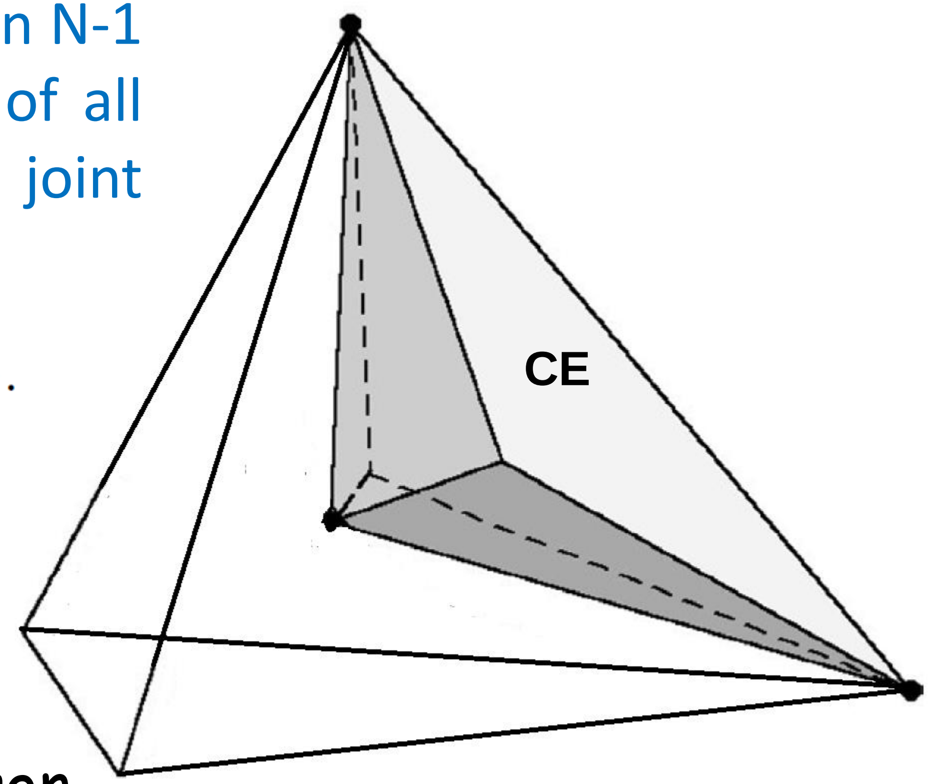
$$\pi(s) \geq 0 \quad \text{for all } s \in S$$

$$\sum_{s \in S} \pi(s) = 1$$

The first two constraints define an N-1 dimensional simplex, consisting of all probability distributions on joint strategies.

$$\sum_{s_{-i} \in S_{-i}} \pi(s)(u_i(s) - u_i(d_i, s_{-i})) \geq, 0 \ \text{ for all } i \text{ and } \quad \text{for all } s_i, d_i \in S_i.$$

A set of linear inequalities (a convex polytope)




- In a 2x2 game, $\Delta(S)$ is a 3-dimensional tetrahedron.
- The figure shows the geometry of the CE of "battle of the sexes".

R. Nau, S. Gomez Canovas, and P. Hansen. On the geometry of nash equilibria and correlated equilibria. *International Journal of Game Theory*, 32(4):443{453, 2004.

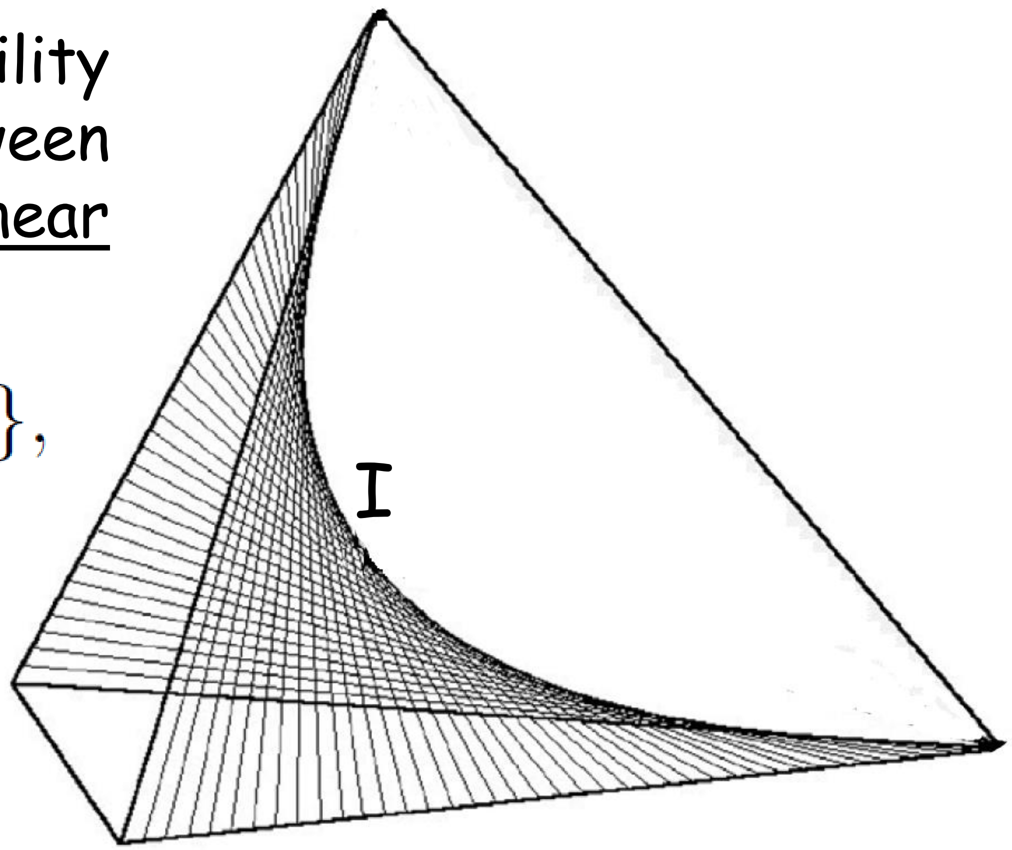
# On the geometry of Nash equilibria and correlated equilibria

- Define I to be the set of all joint probability distributions that are independent between players. I is defined by a system of nonlinear constraints:

$$I = \{\pi(\boldsymbol{a}) = \pi_1(a_1) \times \cdots \times \pi_n(a_n) \quad \forall \boldsymbol{a} \in A\},$$

- I is locally non-convex in the sense that a strictly convex combination of two independent joint distributions in which two or more players have distinct marginal distributions is not independent.

- In a 2x2 game, I is a 2-dimensional saddle.




The figure shows the geometry of I in "battle of the sexes".

R. Nau, S. Gomez Canovas, and P. Hansen. On the geometry of nash equilibria and correlated equilibria. *International Journal of Game Theory*, 32(4):443{453, 2004.

# On the geometry of Nash equilibria and correlated equilibria

- A correlated equilibrium involves a single randomization over action profiles, while in a Nash equilibrium agents randomize separately:

$$\sum_{a \in A} u_i(a) \prod_{j \in N} p_j(a_j) \geq \sum_{a \in A} u_i(a'_i, a_{-i}) \prod_{j \in N \setminus \{i\}} p_j(a_j) \quad \forall i \in N, \forall a'_i \in A_i.$$

- The constraint is nonlinear because of the product $\prod_{j \in N} p_j(a_j)$

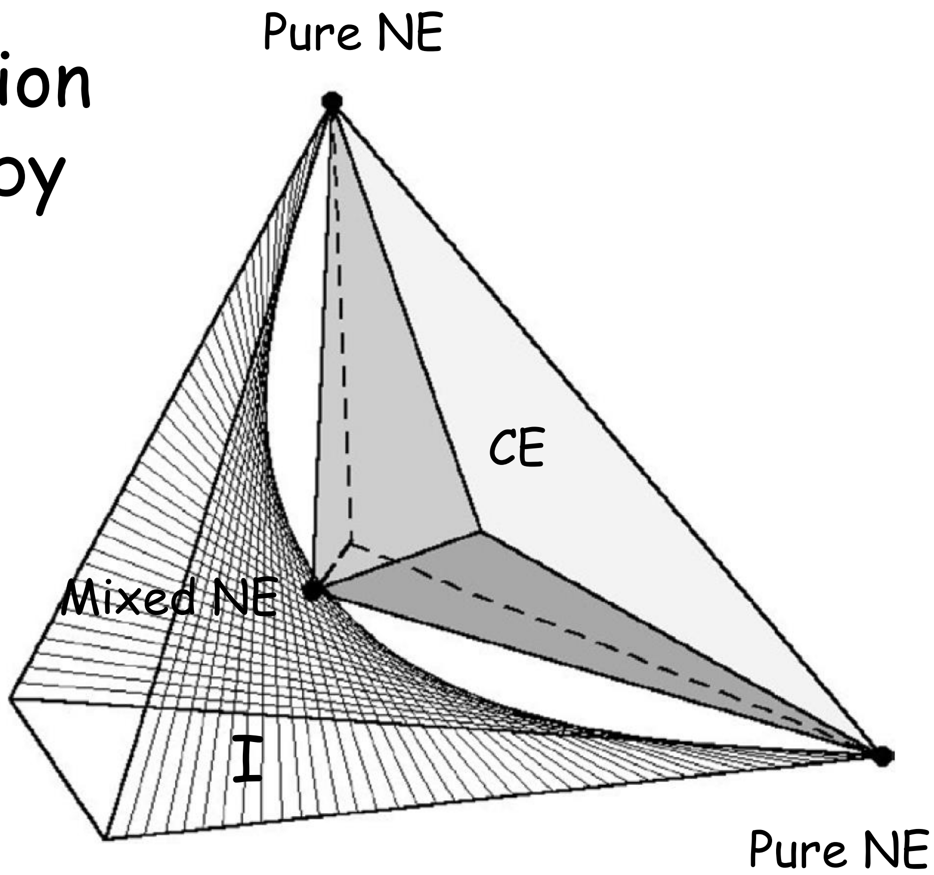
❖ The set of Nash equilibrium distributions may be **nonconvex** or **disconnected**.

❖ Solving for Nash equilibria in games with three or more players may require **nonlinear optimization** or the solution of systems of nonlinear equations.

R. Nau, S. Gomez Canovas, and P. Hansen. On the geometry of nash equilibria and correlated equilibria. *International Journal of Game Theory*, 32(4):443{453, 2004.

- The set of Nash equilibria is the intersection of the set of CE and I, which is nonempty by virtue of Nash's (1951) existence proof.

In general, the set of CE distributions of an n-player non-cooperative game is a convex polytope and the Nash equilibria all lie on the boundary of the polytope.




- The figure shows the geometry of the CE and NE of "battle of the sexes".

R. Nau, S. Gomez Canovas, and P. Hansen. On the geometry of nash equilibria and correlated equilibria. *International Journal of Game Theory*, 32(4):443{453, 2004.

# On the set of achievable payoffs

NE vs. Public CE vs. Private CE

# Expressing NE as CE

- Recall the conditions for $\pi$ to be a correlated equilibrium are that:

$$\sum_{a_{-i}\in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a_i) \geq \sum_{a_{-i}\in A_{-i}} \pi(a_{-i}, a_i) R_i(a_{-i}, a'_i)$$

$$\textit{for all } i \in N, \textit{ for all } a_i \in A_i \textit{ with } \pi(a_i) > 0, \textit{ and for all } a'_i \in A_i$$

- When the "recommendations" are independent across players (i.e., when the joint distribution $\pi$ satisfies:

$$\pi(\boldsymbol{a}) = \pi_1(a_1) \times \cdots \times \pi_n(a_n) \quad \forall \boldsymbol{a} \in \boldsymbol{A}$$

  - in this special case, a correlated equilibrium is just a Nash equilibrium of the game. We will show this more formally in the next slide.

# Any NE is also a CE.

**Proof**. Recall the definition of (mixed) NE. A mixed strategy profile $\boldsymbol{\sigma}^*$ is NE iff for $\forall i, \forall a_i \in A_i$:

$$R_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq R_i(a_i, \boldsymbol{\sigma}_{-i}^*)$$

which is equivalent to say that for all $a_i^* \in supp(\sigma_i^*)$ and $\forall a_i \in A_i$:

$$\sum_{\boldsymbol{a}_{-i}} \underbrace{\boldsymbol{\sigma}_{-i}^*(\boldsymbol{a}_{-i})}_{=\boldsymbol{\sigma}^*(\boldsymbol{a}_{-i}|a_i)} R_i(a_i^*, \boldsymbol{\sigma}_{-i}^*) \geq \sum_{\boldsymbol{a}_{-i}} \underbrace{\boldsymbol{\sigma}_{-i}^*(\boldsymbol{a}_{-i})}_{=\boldsymbol{\sigma}^*(\boldsymbol{a}_{-i}|a_i)} R_i(a_i, \boldsymbol{\sigma}_{-i}^*)$$

Comparing the above inequality with the definition of CE concludes the proof. ❑

# Public CE

- At the other extreme, when the signals are fully correlated (i.e., common, or public—like "sunspots"), each signal must necessarily be followed by a Nash equilibrium play!

- Hence such correlated equilibria—called **publicly correlated equilibria**—correspond to **weighted averages (convex combinations) of Nash equilibria** of a game.

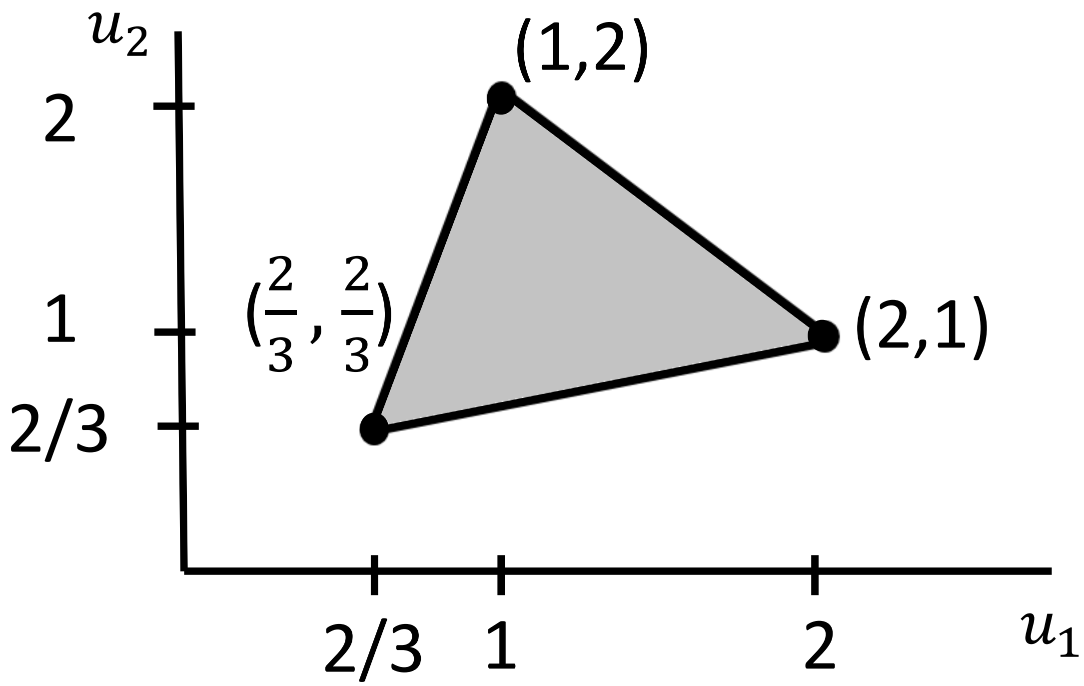
| | HOCKEY | THEATER |
|---|---|---|
| HOCKEY | 2, 1 | 0, 0 |
| THEATER | 0, 0 | 1, 2 |

| | HOCKEY | THEATER |
|---|---|---|
| HOCKEY | 1/2 | 0 |
| THEATER | 0 | 1/2 |

The Battle of the Sexes game (left) and a (publicly) correlated equilibrium (right).

# Public CE (cont'd)

- Public CE: any randomization over Nash equilibria is also a correlated equilibrium and can be attained by joint observation of a public signal.
- It follows that the set of attainable payoffs by a public CE is the convex hull of the payoff vectors given by all NE (the smallest convex set that contains all these vectors).
- This set can be obtained by joining the points given by each NE payoff vector:




| | HOCKEY | THEATER |
|---|---|---|
| HOCKEY | 2, 1 | 0, 0 |
| THEATER | 0, 0 | 1, 2 |

# Private CE

- In general, when the signals are neither independent nor fully correlated, new equilibria arise! For example, in the Chicken game, there is a correlated equilibrium that yields equal probabilities of 1/3 to each action combination except (STAY STAY).

| | LEAVE | STAY |
|---|---|---|
| LEAVE | 5, 5 | 3, 6 |
| STAY | 6, 3 | 0, 0 |

| | LEAVE | STAY |
|---|---|---|
| LEAVE | 1/3 | 1/3 |
| STAY | 1/3 | 0 |

A correlated equilibrium in the Chicken game.

- Indeed, let the signal to each player be L or S; think of this as a recommendation to play LEAVE or STAY, respectively.
- When row gets the signal L, he assigns a (conditional) probability of 1/2 to each one of the two pairs of signals (L L) and (L S);
- So, if column follows his recommendation, then row gets payoff of 4 = (1/2)5 + (1/2)3 from playing LEAVE, and only 3 = (1/2) 6 + (1/2) 0 from deviating to STAY.
- When row gets the signal S, he deduces that the pair of signals is necessarily (S L), so if column indeed plays LEAVE then row is better off choosing STAY. Similarly for the column player.

# Private CE (Cont'd)

2 PNES with payoffs: (6,3), (3,6)

1 MNE $((\frac{3}{4},\frac{1}{4}),(\frac{3}{4},\frac{1}{4}))$ with payoff: (4.5,2.625).

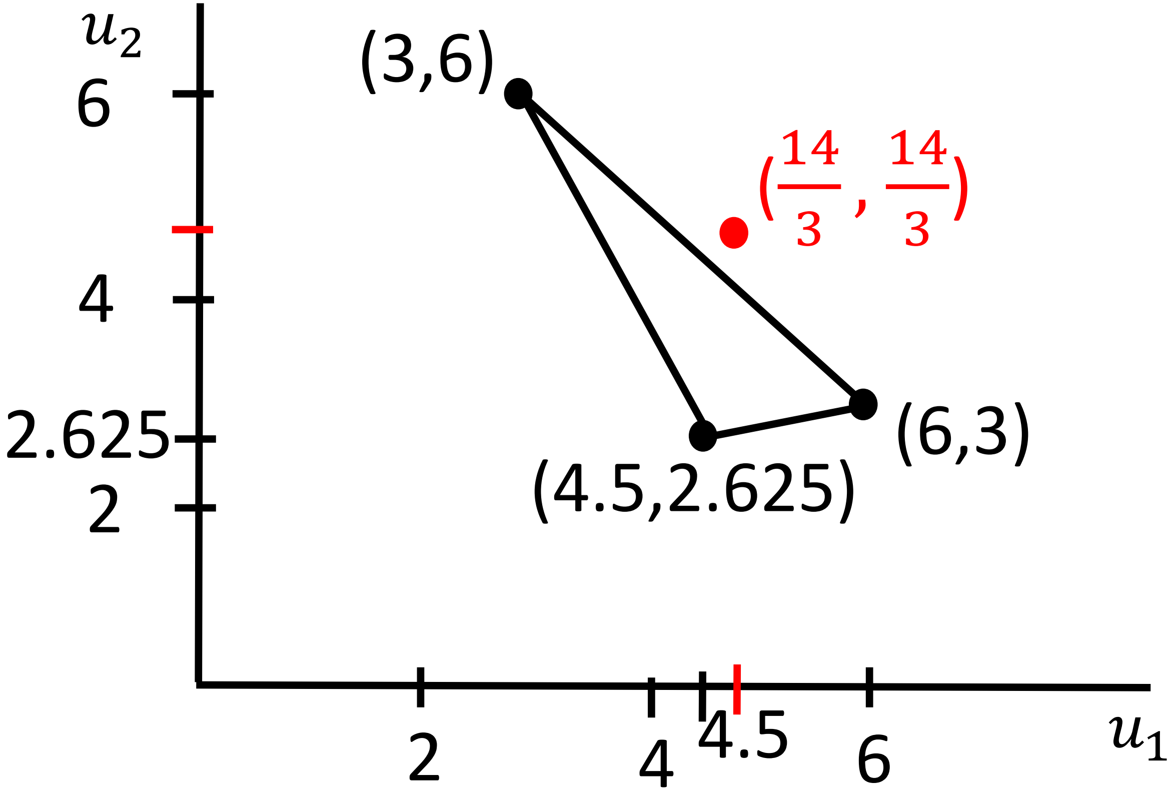
- With perfectly correlated signals (public events), the set of correlated equilibrium payoff profiles is the convex hull of these three points.
- With signals that are imperfectly correlated, new payoffs are possible!

| | LEAVE | STAY |
|---|---|---|
| LEAVE | 5, 5 | 3, 6 |
| STAY | 6, 3 | 0, 0 |

| | LEAVE | STAY |
|---|---|---|
| LEAVE | 1/3 | 1/3 |
| STAY | 1/3 | 0 |

A correlated equilibrium in the Chicken game.

- The payoff profile from Private CE can reach outside the convex hull of pure and mixed strategy NE payoffs.




- More complicated examples could be constructed with correlated equilibrium payoff profiles that **Pareto dominate** pure and mixed strategy NE pay-offs.

# Example Problems

# Example 1: CE Calculation

Consider the following game:

Compute the set of correlated equilibrium distributions.
(It suffices to write down the necessary and sufficient conditions; )

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $a$ | 0,0 | 0,0 | 6,1 |
| $b$ | 0,0 | 3,3 | 0, 0 |
| $c$ | 1,6 | 0,0 | 5,5 |

Recall that for $\pi$ to be CE, we should have:

$$\sum_{s_{-i} \in S_{-i}} \pi(s)(u_i(s) - u_i(d_i, s_{-i})) \geq, 0 \text{ for all } i \text{ and } \quad \text{for all } s_i, d_i \in S_i.$$

The inequalities for player 1 are

$$\pi(a,x)\big(u_1(a,x) - u_1(b,x)\big) + \pi(a,y)\big(u_1(a,y) - u_1(b,y)\big) + \pi(a,z)\big(u_1(a,z) - u_1(b,z)\big) \geq 0$$

$$\pi(a,x)\big(u_1(a,x) - u_1(c,x)\big) + \pi(a,y)\big(u_1(a,y) - u_1(c,y)\big) + \pi(a,z)\big(u_1(a,z) - u_1(c,z)\big) \geq 0$$

$$\pi(b,x)\big(u_1(b,x) - u_1(a,x)\big) + \pi(b,y)\big(u_1(b,y) - u_1(a,y)\big) + \pi(b,z)\big(u_1(b,z) - u_1(a,z)\big) \geq 0$$

$$\pi(b,x)\big(u_1(b,x) - u_1(c,x)\big) + \pi(b,y)\big(u_1(b,y) - u_1(c,y)\big) + \pi(b,z)\big(u_1(b,z) - u_1(c,z)\big) \geq 0$$

$$\pi(c,x)\big(u_1(c,x) - u_1(a,x)\big) + \pi(c,y)\big(u_1(c,y) - u_1(a,y)\big) + \pi(c,z)\big(u_1(c,z) - u_1(a,z)\big) \geq 0$$

$$\pi(c,x)\big(u_1(c,x) - u_1(b,x)\big) + \pi(c,y)\big(u_1(c,y) - u_1(b,y)\big) + \pi(c,z)\big(u_1(c,z) - u_1(b,z)\big) \geq 0$$

# Example 1: CE Calculation (cont'd)

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $a$ | 0,0 | 0,0 | 6,1 |
| $b$ | 0,0 | 3,3 | 0, 0 |
| $c$ | 1,6 | 0,0 | 5,5 |

The inequalities for player 1 are

$$
\begin{aligned}
6\pi(a,z) &\geq 3\pi(a,y) \\
6\pi(a,z) &\geq \pi(a,x) + 5\pi(a,z) \\
3\pi(b,y) &\geq 6\pi(b,z) \\
3\pi(b,y) &\geq \pi(b,x) + 5\pi(b,z) \\
\pi(c,x) + 5\pi(c,z) &\geq 6\pi(c,z) \\
\pi(c,x) + 5\pi(c,z) &\geq 3\pi(c,y)\,,
\end{aligned}
$$

# Example 1: CE Calculation (cont'd)

Similarly, the conditions for player 2 are

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $a$ | 0,0 | 0,0 | 6,1 |
| $b$ | 0,0 | 3,3 | 0, 0 |
| $c$ | 1,6 | 0,0 | 5,5 |

$$\begin{aligned}
2\pi(c,x) &\geq \pi(b,x) \\
\pi(c,x) &\geq \pi(a,x) \\
\pi(b,y) &\geq 2\pi(c,y) \\
3\pi(b,y) &\geq \pi(a,y) + 5\pi(c,y) \\
\pi(a,z) &\geq 2\pi(c,z) \\
\pi(c,x) + 5\pi(c,z) &\geq 3\pi(b,z)\,.
\end{aligned}$$

# Example 2: Characterizing CE Payoffs

Identify a correlated equilibrium payoff vector that is not in the convex hull of Nash equilibrium payoff vectors.

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $a$ | 0,0 | 0,0 | 6,1 |
| $b$ | 0,0 | 3,3 | 0, 0 |
| $c$ | 1,6 | 0,0 | 5,5 |

Take $p(a,z) = p(c,x) = p(c,z) = 1/3$, which yields the payoff vector of $(4,4)$.

- This is clearly outside of the convex hull of Nash equilibria, as the sum of the payoffs are at most 7 in every Nash equilibrium.
  - To see this, note that in order for sum to be greater than 7 in any Nash equilibrium $\sigma$, it must be that $\sigma_1(c)\sigma_2(z) > 0$.

For $c$ to be a best response, we must have

$$\sigma_2(x) + 5\sigma_2(z) \geq 6\sigma_2(z), \text{ i.e., } \sigma_2(x) \geq \sigma_2(z).$$

Similarly, we must have $\sigma_1(a) \geq \sigma_1(c)$.

Hence, $\sigma_1(a)\sigma_2(x) > \sigma_1(c)\sigma_2(z)$.

We saw that in order for (c,z) to be played at an NE, we should have:

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $a$ | 0,0 | 0,0 | 6,1 |
| $b$ | 0,0 | 3,3 | 0, 0 |
| $c$ | 1,6 | 0,0 | 5,5 |

$$\sigma_1(a)\,\sigma_2(x) > \sigma_1(c)\,\sigma_2(z)$$

Thus, the total payoff from (a,x) and (c,z) cannot be higher than 10/2 = 5.

Since the sum in all other profiles is less than or equal to 7, the sum of Nash equilibrium payoffs is less than or equal to 7.